A Study of $W_2$ - Curvature Tensor of a $N(k)$ - Quasi Einstein Manifold

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ABSTRACT

In this paper, the study of $W_2$-curvature tensor in $N(k)$-quasi Einstein manifold satisfying $W_2(\xi, X).W_2 = 0$ is carried out.


Keywords : $W_2$ - Curvature tensor, Einstein manifolds, quasi Einstein manifolds, $N(k)$ - quasi Einstein manifolds.

I. INTRODUCTION

A NON-FLAT n-dimensional Riemannian manifold $(M, g)$ is said to be a quasi Einstein manifold [2] if its Ricci tensor satisfies

$$S(X, Y) = a g(X, Y) + b \eta(X) \eta(Y)$$

for all $X, Y \in TM$. Where $a$ and $b \neq 0$ are smooth functions and $\eta$ be a nonzero 1-form such that

$$g(X, \xi) = \eta(X), \quad \eta(\xi) = 1$$

for the associated vector field $\xi$, which is equivalent to

$$Q = aI + b \eta \otimes \xi$$

The 1-form $\eta$ is called the associated 1-form and the unit vector field $\xi$ is called the generator of the manifold. If the generator $\xi$ belongs to k-nullity distribution $N(k)$ then the quasi Einstein manifold is called as an $N(k)$-quasi Einstein manifold [8].

In [8], it was proved that a conformally flat quasi-Einstein manifold is $N(k)$-quasi Einstein. Consequently, it was shown that a 3-dimensional quasi-Einstein manifold is an $N(k)$-quasi - Einstein manifold. The derivation conditions, $R(\xi, X) \cdot R = 0$ and $R(\xi, X) \cdot S = 0$ were also studied, where $R$ and $S$ denote the curvature and Ricci tensor respectively.

On the other hand in [1] the derivative conditions $Z(\xi, X).Z = 0, Z(\xi, X).R = 0$ and $R(\xi, X).Z = 0$ on contact metric manifolds are studied, where $Z$ is concircular curvature tensor. In [7], the condition $X(\xi, X).S = 0$ is studied. In [5], $N(k)$-quasi Einstein manifolds satisfying the conditions $R(\xi, X).W_2 = 0, W_2(\xi, X), S = 0, P(\xi, X), W_2 = 0$ where $P$ denotes the projective curvature tensor are studied. In this paper, I study the derivation conditions $W_2(\xi, X), W_2 = 0$ on an $N(k)$-quasi Einstein manifold. The paper is organized as follows : Section 2 contains necessary details about $N(k)$-quasi Einstein manifolds and the $W_2$- curvature tensor. In section 3 the conditions $W_2(\xi, X), W_2 = 0$ on an $N(k)$-quasi Einstein manifold is studied.

II. PRELIMINARIES

Let $M$ be a $(2n + 1)$ dimensional Riemannian manifold. The $W_2$-curvature tensor [4] is defined as

$$W_2(X, Y) = R(X, Y)Z - \frac{1}{2n} \nabla \frac{b(X, Y)QX}{g(X, Y)QY}$$

Where $R$ is the Riemannian curvature tensor and $\nabla$ is the covariant derivative.
Q is the Ricci operator defined as
\[ S(X,Y) = g(QX,Y), \quad X,Y \in TM \] (5)

Equation (4) can also be written as
\[ W_2(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{2n} \{ g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \} \] (6)
for all \( X,Y,Z \in TM \).

The k-nullity distribution \( N(k) \) [6] of a Riemannian manifold \( M \) is defined by
\[ N(k) : p \rightarrow N_p(k) = \{ Z \in T_pM : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y) \} \]
for all \( X,Y \in TM \), where \( k \) is some smooth function.

A. Definition [8]

The \( N(k) \)-quasi Einstein manifold is defined as:

Let \((M^{2n+1}; g)\) be a quasi Einstein manifold. If the generator \( \xi \) belongs to the k-nullity distribution \( N(k) \) for some smooth function \( k \), then we say that \((M^{2n+1}; g)\) is an \( N(k) \)-quasi Einstein manifold.

B. Lemma [3]

In an \((2n+1)-\)dimensional \( N(k) \)-quasi Einstein manifold it follows that
\[ k = \frac{a+b}{2n} \] (7)

From equations (1) and (2) it follows that
\[ S(X,\xi) = (a+b)\eta(X), \quad X \in TM. \]

\[ r = (2n+1)a+b \] (8)

where \( r \) is the scalar curvature of \( M \).

Since
\[ R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) \] (9)

Using (7) in above equation we get
\[ (R(X,Y)\xi = \frac{a+b}{2n} \{ \eta(Y)X - \eta(X)Y \} \] (10)
which is equivalent to
\[ R(X,\xi)Y = \frac{a+b}{2n} \{ \eta(Y)X - g(X,\xi)Y \} \]
\[ = -R(\xi,X)Y \] (11)

Taking \( X = \xi \) and \( Y = X \) in equation (10) we get
\[ R(\xi,X)\xi = \frac{a+b}{2n} \{ \eta(X)\xi - X \} \] (12)

C. Proposition

In an \((2n+1)-\)dimensional \( N(k) \)-quasi Einstein manifold, the \( W_2 \) curvature tensor satisfies
\[ W_2(X,Y)\xi = \frac{b}{2n} (\eta(Y)X - \eta(X)Y) \] (13)

\[ W_2(\xi,X)\xi = \frac{b}{2n} (Y - \eta(Y)\xi) = -W_2(X,X)\xi \] (14)

\[ \eta(W_2(X,Y)\xi) = 0 \] (15)

\[ W_2(\xi,Y)Z = -\frac{b}{2n} (Y - \eta(Y)\xi)\eta(Z) \] (16)

\[ \eta(W_2(\xi,Y)Z) = 0 \] (17)

Proof - where \( N(k) \)-quasi

D. Theorem

Let \( M \) be \((2n+1)\)-dimensional \( N(k) \)-quasi Einstein manifold. If \( M \) satisfies the condition \( W_2(\xi,X)W_2 = 0 \) then \( M \) is Einstein and \( a = 0 \).

Proof:

Let \( W_2(\xi,X) \cdot W_2 = 0 \), this implies
\[ 0 = [W_2(\xi,U),W_2(X,Y)]\xi - W_2(W_2(\xi,U)X,Y)\xi - W_2(X,W_2(\xi,U)Y)\xi \]

\[ = -W_2(\xi,U)W_2(X,Y)\xi - W_2(X,Y)W_2(\xi,U)\xi - W_2(W_2(\xi,U)X,Y)\xi - W_2(X,W_2(\xi,U)Y)\xi \]

Using equations (14), (15) and (16) in above equation we get
\[ W_2(X,Y)Z = \frac{b}{2n} (\eta(Y)X - \eta(X)Y)\eta(Z). \]

Using (4) in above equation we get
\[ R(X,Y,Z,W) = \frac{b}{2n} (g(X,W)\eta(Z) - g(Y,W)\eta(X)\eta(Z)) \]
\[ + \frac{1}{2n} (g(Y,Z)S(X,W) - g(X,Z)S(Y,W)) \]

Contracting above equation and using eq. (1) and (8) we get
\[ \frac{b}{2n} (g(Y,Z) - \eta(Y)\eta(Z)) = 0 \]
But $b \neq 0$, we get 

$$g(Y, Z) = \eta(Y)\eta(Z)$$

Using above equation in (1) we get 

$$S(X, Y) = (a + b)g(X, Y).$$

This equation shows that manifold is Einstein. Contacting above equation and using (8) we get $a = 0$

**Corollary:** 
Let $M$ be $(2n + 1)$ dimensional $N(k)$ quasi Einstein manifold. If $M$ satisfies the condition

$$W_2(\xi, X) \cdot W_2 = 0.$$ 

Then 

$$R(X, Y)Z = \frac{b}{2n}\{\eta(Y)X - \eta(X)Y\}\eta(Z),$$

$x, y, z \in TM$

### III. CONCLUSION

A $(2n + 1)$-dimensional $N(k)$-quasi Einstein manifold $M$ is Einstein if it satisfies the conditions of $W_2(\xi, X)W_2 = 0$.

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